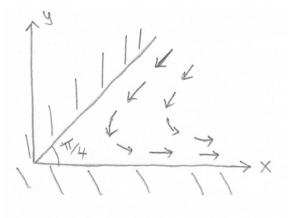
SNAP 2017. Laplace's equation and conformal maps.

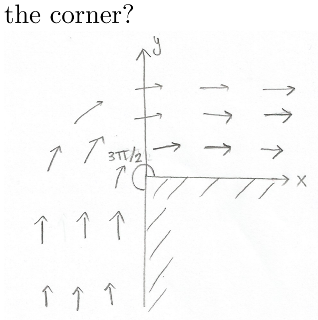
Problem Set 3

1. In lecture, we found a complex potential for a fluid flow around a corner of angle $\pi/2$.

   (a) Find a complex potential for the fluid flow when the corner has angle $\pi/4$ as shown. What is the speed of the fluid flow as you approach the corner?

   (b) Find a complex potential for the fluid flow when the corner has angle $3\pi/2$ as shown. What is the speed of the fluid flow as you approach the corner?

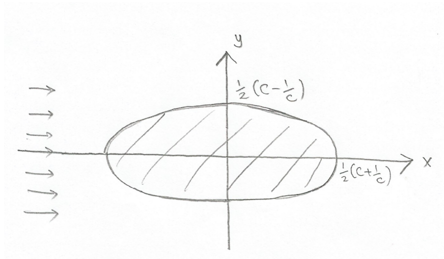
   

   Solution. (a) For angle $\pi/4$ apply $w = z^4$ to map to the upper half plane, where a complex potential is given by $f = aw$ for $a > 0$. Hence the complex potential is $f(z) = az^4$. The speed of the fluid tends to zero at the corner (easier to compute in polar coordinates). (b) For angle $3\pi/2$ the complex potential is $f(z) = az^{2/3}$. The speed tends to infinity at the corner.

2. Suppose a uniform horizontal flow of speed V encounters a cylindrical obstacle whose cross section is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a = \frac{1}{2}(c + 1/c), \ b = \frac{1}{2}(c - 1/c),$$

   for a constant $c > 1$. Find the complex potential describing the flow.



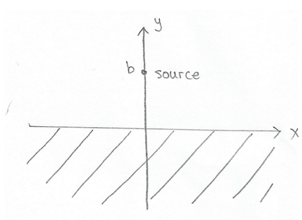
Hint: first consider the cylinder with circular cross section of radius c

Solution. Apply the map $w = z + \sqrt{z^2 - 1}$ which sends the ellipse to a circle of radius c. The cylinder with circular cross section of radius c in the w plane has complex potential $f(w) = V_0(w + c^2/w)$, for a constant V_0, so we get the potential

$$f(z) = V_0 \left(z + \sqrt{z^2 - 1} + \frac{c^2}{z + \sqrt{z^2 - 1}} \right),$$

and since the speed is V we get $V_0 = V/2$.

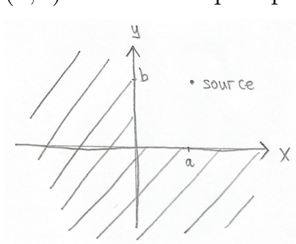
3. Suppose a fluid flow in the upper half plane is given by a source at a distance b as shown. Find a complex potential for the fluid flow.



Hint: consider a fluid flow on the whole complex plane. Add a "mirror" source.

Solution. We add a source at a distance b on the other side of the wall. Get $f(z) = C(\log(z - bi) + \log(z + bi))$ for a constant $C > 0$.

4. Suppose a fluid flow in the first quadrant is given by a source located at (a, b). Find a complex potential for the fluid flow.

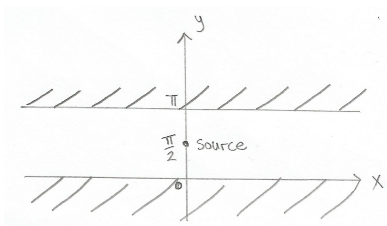


Solution. Add mirror sources at $(-a, b)$, $(-a, -b)$ and $(a, -b)$, to get

$$f(z) = C\left(\log(z - (a+bi)) + \log(z - (-a+bi)) \right.$$
$$\left. + \log(z - (-a-bi)) + \log(z - (a-bi))\right),$$

for a constant $C > 0$.

5. Consider a fluid source at the point $(0, \pi/2)$ in a infinite channel of height π as shown. Determine the complex potential for the fluid flow.



Hint: use a conformal map

Solution. Use $z \mapsto e^z$ to map the channel to the upper half plane with a single source at $z = i$. From problem 3 we get $f(z) = C(\log(e^z - i) + \log(e^z + i))$ for a constant $C > 0$.

6. Fix a complex number β and a real number $r_0 > 0$. Suppose that the image of the disk $|z - \beta| \le r_0$ under the Joukowsky map $w(z) = \frac{1}{2}(z + 1/z)$ defines an airfoil shaped domain in the complex plane which contains the set $[-1, 1] \times \{0\}$. Show that the complex potential for the horizonal flow past the airfoil is a constant multiple of

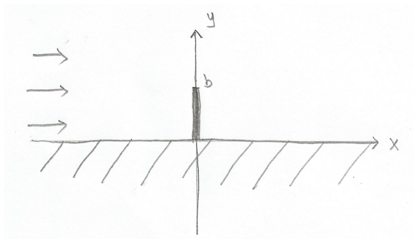
$$f(w) = \frac{w - \beta + \sqrt{w^2 - 1}}{r_0} + \frac{r_0(w - \beta - \sqrt{w^2 - 1})}{\beta^2 + 1 - 2\beta w}.$$

Solution. Consider the unit disk in the ζ plane, where a horizontal fluid flow is given by $\frac{1}{2}(\zeta + \frac{1}{\zeta})$. Map via $z = r_0\zeta + \beta$ to the disk $|z - \beta| \le r_0$. The complex potential of a horizontal fluid flow in the z plane is $f(z) = \frac{1}{2}\left(\frac{z-\beta}{r_0} + \frac{r_0}{z-\beta}\right)$. We map to the w plane via $w(z) = \frac{1}{2}(z + 1/z)$ which has inverse $z = w + \sqrt{w^2 - 1}$. Then the complex potential in the w-plane is, up to a constant multiple,

$$f(w) = \frac{w + \sqrt{w^2 - 1} - \beta}{r_0} + \frac{r_0}{w + \sqrt{w^2 - 1} - \beta}$$

and we get the required answer after simplifying.

7. Suppose a uniform horizontal flow of speed V encounters a hurdle of height b and negligible width. Find the complex potential describing the flow.



Solution. Consider the horizontal flow on the whole z plane with a plate of vertical thin plate of length $2b$. We are now more or less in the situation considered in lecture. Apply the map $w = iz/b$ which sends the vertical plate to a horizontal one $[-1, 1] \times \{0\}$ (the flow is now vertical). The inverse of the Joukowsky map $\zeta = w + \sqrt{w^2 - 1}$ sends this to a vertical flow around a circular obstacle and finally rotate by $\eta = -i\zeta$ to get the horizontal flow around a circular obstacle which has complex potential $V_0(\eta + 1/\eta)$ for some $V_0 > 0$. Converting back to the z coordinate we get

$$f(z) = V_0 \left(\frac{z}{b} - i\sqrt{-z^2/b^2 - 1} + \frac{i}{iz/b + \sqrt{-z^2/b^2 - 1}} \right).$$

Finally, by considering $|z|$ large we get $V = V_0/b$ so $V_0 = Vb$.